Math 245B Lecture 23 Notes

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1 Translation Operators and Relationships Between L^p Spaces

1.1 Translation operators on L^p spaces

Let m be Lebesgue measure on $\mathbb{R}^d$, and let $t \in \mathbb{R}^d$. Let $\tau_t : \mathbb{R}^d \to \mathbb{R}^d$ send $v \mapsto v - \tau$. This is translation by t, and Lebesgue measure is translation invariant.

Lemma 1.1. *The map T_t sending $f \mapsto f \circ \tau_t$ is an isometry $L^p(m) \to L^p(m)$ for all p.*

However, the functions T_t are not kernel operators.

Lemma 1.2. *Let $p < \infty$. Let $(t_n)_n$ in $\mathbb{R}^d$ be such that $t_n \to 0$. Then $T_{t_n} \to \mathrm{Id}$ on $L^p(m)$ in the strong operator topology but not in $\|\cdot\|_{\mathrm{op}}$.*

Proof. $C_c(\mathbb{R}^d)$ is dense in $L^p(m)$. Let $f \in L^p$. Suppose first that $f \in C_c(\mathbb{R}^d)$. Pick R such that $f|_{B_R^c} = 0$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z-y| < \delta \implies |f(x)-f(y)| < \varepsilon$. If $|t_n| < \delta$, then

$$\begin{aligned} \|T_{t_n} f - f\|_p^p &= \int_{\mathbb{R}^d} |f(x - t_n) - f(x)|^p \, dm(x) \\ &= \int_{B_{R+1}} |f(x - t_n) - f(x)|^p \, dm(x) \\ &\le \varepsilon^p m(B_{R+1}) \\ &\xrightarrow{t_n \to 0} 0. \end{aligned}$$

Similarly, the map $\mathbb{R}^d \to \mathcal{L}(L^p(m), L^p(m))$ sending $t \mapsto T_t$ is continuous from $\mathbb{R}^d$ to the strong operator topology. For general $f \in L^p(m)$, let $\varepsilon > 0$. Choose $g\ inC_c(\mathbb{R}^d)$ such that $\|f - g\|_p < \varepsilon/3$. Choose n large enough such that $\|T_{t_n} g - g\|_p < \varepsilon/3$. Put together,

$$\begin{aligned} \|T_{t_n} f - f\|_p &\le \|T_{t_n}(f - g)\| + \|T_{t_n} g - g\|_p + \|f - g\|_p \\ &\le \|f - g\| + \|T_{t_n} g - g\|_p + \|f - g\|_p \\ &< \varepsilon. \end{aligned}$$

Now let's show that this convergence does not occur in the norm topology. For any $t \neq 0$, there exist $f \in C_c(\mathbb{R}^d$ such that $\|f\|_p = 1$ and $f|_{B_{t/2}^c} = 0$. Then

$$\|T_t f - f\|_p = 2^{1/p}\|f\|_p. \qquad \square$$

1.2 Relationships between L^p spaces

What is the relationship between L^p spaces for different p?

Example 1.1. Look at $((0,\infty)), \mathcal{B}_{(0,\infty)}, m)$. Let $1 \leq p < q < \infty$. Let $f_a(x) = x^{-a}$ for some choice of $a > 0$. Observe:

1. The function $f_a \mathbb{1}_{(0,1)} \in L^p$ iff $p < 1/a$.
2. The function $f_a \mathbb{1}_{(1,\infty)} \in L^p$ iff $p > 1/a$.

So $L^p \setminus L^q \neq \varnothing$, and $L^q \setminus L^p \neq \varnothing$.

Proposition 1.1. *If $0 < p < q < r \leq \infty$, then $L^q \subseteq L^p + L^r$.*

Proof. Let $f \in L^q$. Write $f = f\mathbb{1}_{\{|f|>1\}} + f\mathbb{1}_{\{|f|\leq 1\}}$. Then

$$\|f\mathbb{1}_{\{|f|>1\}}\|_p^p = \int_{\||f|>1\}} |f|^p \, d\mu \leq \int_{\||f|>1\}} |f|^q \, d\mu \leq \int |f|^q \, d\mu = \|f_q\|^q < \infty.$$

The same holds for $f\mathbb{1}_{\{|f|\leq 1\}}$. $\square$

Proposition 1.2. *If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subseteq L^q$, and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $1/q = \lambda(1/p) + (1-\lambda)(1/r)$.*

Proof. It suffices to prove the inequality.

$$\int |f|^q \, d\mu = \int |f|^{\lambda q} |f|^{(1-\lambda)q} \, d\mu$$

Use Hölder's inequality, where $1/s + 1/t = 1$. We will pick the values of s, t later to make sure they work out.

$$\leq \left(\int |f|^{\lambda q s} \, d\mu \right)^{1/s} \left(\int |f|^{(1-\lambda)qt} \, d\mu \right)^{1/t}$$

Pick $s = p/(\lambda q)$ to make things work out as stated in the theorem.

$$\leq \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q}. \qquad \square$$

There is, however, a case where the tails of functions do not count.

Lemma 1.3. *If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p \supseteq L^q$. In particular $\|f\|_p \leq \|f\|_q \mu(X)^{1/p-1/q}$.*

Proof. Let $f \in L^q$. Then, by Hölder's inequality,

$$\|f\|_p^p = \int |f|^p \mathbb{1}_X \, d\mu \leq \|f\|_q (\mu(X))^{1/p-1/q}. \qquad \square$$

Lemma 1.4. *Let A be any set. Let $\ell^p(A) = L^p(A, \mathscr{P}(A), \#)$. Then $\ell^p \subseteq \ell^q$.*

Proof. If $q = \infty$, then

$$\sup_\alpha |f(\alpha)| = (\sup_\alpha |f(\alpha)|^p)^{1/p} \leq \left(\sum_\alpha |f(\alpha)|^p \right)^{1/p} = \|f\|_p.$$

If $p < q < \infty$, then by the previous lemma,

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_\infty^{1-\lambda} \leq \|f\|_p^{\lambda+1-\lambda} = \|f\|_p. \qquad \square$$

1.3 Distribution functions

Fix $(X, \mathcal{M}, \mu)$, and let $f : X \to \mathbb{C}$ be measurable.

Definition 1.1. The **distribution function** of f, $\lambda_f : (0, \infty) \to [0, \infty]$, is

$$\lambda_f(\alpha) = \mu(\{|f| > \alpha\}).$$

Proposition 1.3. *Let λ_f be the distribution function of f.*

1. *λ_f is non-increasing and right-continuous.*
2. *If $|f| \leq |g|$, then $\lambda_f \leq \lambda_g$.*
3. *If $|f_n| \uparrow |f|$, then $\lambda_{f_n}(\alpha) \uparrow \lambda_f(\alpha)$.*
4. *If $f = g + h$, then $\lambda_f(\alpha) \leq \lambda_g(\alpha/2) + \lambda_h(\alpha/2)$.*